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NONHOMOGENEOUS ELASTIC ROD PENETRATORS

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number) The problem of determining the distribution of stress, strain, particle velocity and displacement in long projectiles being impacted is presented. Closed form solutions for these quantities in projectiles with certain types of variable modulus, density and crossectional area are presented. It is concluded that the wave propagation features (including possible failure locations), in inhomogeneous projectiles are somewhat different from those of

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ABSTRACT (Cont'd)

homogeneous projectiles. Applications to other munitions components such as armor and cartridge cases are presented.

TABLE OF CONTENTS

List of Tables

List of Illustrations

NOMENCLATURE

a	:	$[1-(\alpha+k)]/2$, Equation (22)
a	:	function, Equation (32)
a	:	function, Equation (33)
a	:	cross sectional area, Equation (1)
A	:	$\bar{A}/\bar{A}_0$, Equation (8)
$\bar{c}$	:	variable bar wave speed, Equation (4)
c	:	$\bar{c}/\bar{c}_0$, Equation (8)
$\bar{E}$	:	Young's modulus
E	:	$\bar{E}/\bar{E}_0$, Equation (8)
$H(t)$	:	Heavyside Step Function, Equation (7)
k	:	constant; area parameter, Equation (20)
n	:	constant, Equation (23)
p	:	Laplace transform parameter, Equation (21)
s	:	$(\alpha-\lambda)$constant; wave parameter, Equation (19)
$\bar{t}$	:	time, Equation (1)
t	:	$\bar{t}\ \bar{c}_0/\bar{x}_0$, Equation (8)
t_a	:	arrival time of wave, Equation (34)
t_∞	:	constant, Equation (35)
$\bar{u}$	:	axial displacement, Equation (1)
u	:	$\bar{u}\,\bar{E}_{\xi}/\bar{\sigma}_0\bar{X}_0$, Equation (8)
U	:	transform of displacement, Equation (21)
$\frac{dU}{dx}$	:	transform of strain, Equation (26)
—	:	transform of particle velocity, Equation (27)
$\bar{x}$	:	axial coordinate, Equation (1)
x	:	$\bar{x}/\bar{x}_0$, Equation (8)

α : constant; modulus parameter, Equation (15)

β : $[2-\rho]/\alpha$, Equation (23)

λ : constant density parameter, Equation (16)

$\bar{\rho}$: mass density, Equation (1)

ρ : $\bar{\rho}/\bar{\rho}_c$, Equation (8)

$\bar{\sigma}$: axial stress, Equation (1)

σ : $\bar{\sigma}/\bar{\sigma}_o$, Equation (8)

$\bar{\sigma}_o$: constant, Equation (6)

Σ : transform of stress, Equation (21)

$\frac{d\bar{U}}{d\bar{x}}$: strain, Equation (2)

The subscript "o" denotes quantities evaluated x_o

INTRODUCTION

The objective of the work presented here is to determine mathematically the elastic structural response as regards to stress, strain, particle velocity and displacements in long rod projectiles subjected to at one end to impact loading. This study is limited to rod penetrators with a continuously varying inhomogeneity.

The dynamic structural response of projectiles subjected to impact loading from a target is an extremely important factor in projectile design. Generally, at ballistic velocities, the response is best described by a model accounting for dynamic plastic and, possibly, hydrodynamic strains in addition to the smaller elastic strains. However, at very low impact velocities a reasonable approximation is to view the projectile response as totally elastic. This will be the assumption followed in the present analysis. Representative material properties - such as elastic modulus, mass density and yield strength, may vary within the projectile. It is the purpose of this study to present theoretical closed form solutions of the governing differential equations which include the effects of such variations or inhomogeneities. These solutions, while of limited practical value, are principally intended for use in evaluating approximate or numerical methods which are designed to aid in the solution of realistic problems. As will be shown, the specific forms of inhomogeneity used allow a wide range of material property variations.

The propagation of elastic strain waves in inhomogeneous has been extensively investigated in recent years.* This is largely due to the fact that it is now well established that nonuniform thermal gradients (produced, perhaps, during the manufacturing process) will in turn produce a variable modulus and density in a solid or

---

* See References 1 through 15.

hollow bar[1 to 5] or in a cylinder.[5, 6] The propagation of elastic plastic waves in thin inhomogeneous bars was considered by Cristescu,[7] and Perzyna.[8, 9]

In the present analysis it is assumed that the projectile is an elastic solid bar of variable cross sectional area using the classical, or Love, bar theory.[16] It is also assumed that the bar strikes the target at zero degrees obliquity and thus that the stress pulse is always normal to the bar axis. The use of the classical bar theory will not produce serious error if the rate of change of the cross sectional area is small and the propagated waves have wavelengths large compared to the dimensions of the cross section.[16] The modulus of elasticity and density will vary continuously along the bar in the form of a power law in the spacial coordinate. Such representation has proved valuable to earlier investigators of plane waves in inhomogeneous rods,[1, 2, 5, 10] or shear waves in infinite media.[17, 18]

Earlier investigators have treated this problem in several ways ranging from purely analytical to wholly numerical. Closed form solutions have been obtained by Reiss,[13] Datta,[14] Sur,[15] Cooper,[20] Reddy,[23] Nayfeh and Nemat-Nasser.[24] Datta and Sur obtained solutions in which the modulus varies, respectively, as a linear or exponential function of the space variable and the rod was finite. Reiss[13] has presented a class of closed solutions to seven problems having inhomogeneous properties by reducing all governing equations to several canonical forms having easily obtainable solutions. No specific examples are given. The technique presented here and that of Reiss[13] produce identical results in some circumstances. Cooper[20] has used the method of progressing wave expansions to obtain solutions in the form of an infinite series. For a specific power law inhomogeneity in the modulus the series terminates in one term. This single solution is of the form obtained in the present report. The latter comment applies as well to another approach taken by Reddy.[23] Nayfeh and Nemat-Nasser[24] have presented a large class of closed form solutions as well, but only for time harmonic waves.

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*See References 1 through 24.

THEORY

Governing Equations

Consider an isotropic but continuously inhomogeneous projectile (or bar) of elastic material, and neglect completely the Poission effect, radial inertia and bending. Thus the equilibrium equation for the linear elastic projectile is:

$$\frac{\partial \bar{\sigma}}{\partial \bar{x}} + \frac{1}{\bar{A}} \frac{\partial \bar{A}}{\partial \bar{x}} \bar{\sigma} = \bar{\rho} \frac{\partial^2 \bar{u}}{\partial \bar{t}^2} ; \quad \bar{u} = \bar{u}(\bar{x}, \bar{t}) \tag{1}$$

where $\bar{\sigma}$ = the only non-vanishing stress component

$\bar{A}$ = the(variable)cross section area

$\bar{\rho}$ = the(variable)mass density

$\bar{u}$ = the displacement

$\bar{x}, \bar{t}$ = the spacial and temporal coordinates

To this is added the stress-displacement relation:

$$\bar{\sigma} = \bar{E} \frac{\partial \bar{u}}{\partial \bar{x}} \tag{2}$$

where $\bar{\bar{E}}$ is the modulus and $\frac{\partial \bar{u}}{\partial \bar{x}}$ is the only non-vanishing strain component.

Eliminating the stress between Equations (1) and (2) yields the Navier type relation:

$$\frac{\partial^2 \bar{u}}{\partial \bar{x}^2} + \frac{\partial \bar{u}}{\partial \bar{x}} \left[\frac{E'}{E} + \frac{A'}{A} \right] = \frac{1}{(\bar{c})^2} \frac{\partial^2 \bar{u}}{\partial \bar{t}^2} \tag{3}$$

where $\bar{c}^2 = \bar{E}/\bar{\rho}$ (4)

and a (') denotes the spacial derivative.

The problem under consideration is that of an infinitely long elastic projectile, motionless prior to $\bar{t} = 0$. Thus the appropriate initial conditions subsidary to Equation (4) are (Figure 1)

$$\bar{u} = \frac{\partial \bar{u}}{\partial \bar{t}} = 0 \quad \text{on} \quad \bar{t} \leq 0 \; ; \; \bar{x}_o \leq \bar{x} \leq \infty \tag{5}$$

For a step input in stress at $\bar{x}_o$, the boundary condition is

$$\bar{E} \frac{\partial \bar{u}}{\partial \bar{x}} = \bar{\sigma}_c H(\bar{t}) \qquad \text{at} \quad \bar{x} = \bar{x}_o \tag{6}$$

where $\bar{\sigma}_c$ is a constant amplitude and $H(\bar{t})$ is the Heaviside Step Function given as

$$\begin{aligned} H(\bar{t}) &= 0 \quad \text{for} \quad \bar{t} < 0 \\ H(\bar{t}) &= 1 \quad \text{for} \quad \bar{t} > 0 \end{aligned} \tag{7}$$

Transforming to dimensionless variables via the following definitions:

$$\begin{aligned} x &= \bar{x}/\bar{x}_o & \sigma &= \bar{\sigma}/\bar{\sigma}_o \\ E &= \bar{E}/\bar{E}_c & u &= \bar{u}\bar{E}_o/\bar{\sigma}_o\bar{x}_o \\ c &= \bar{c}/\bar{c}_o & A &= \bar{A}/\bar{A}_o \\ t &= \bar{t}\bar{c}_o/\bar{x}_o & \rho &= \bar{\rho}/\bar{\rho}_o \\ & & c_o &= \sqrt{\bar{E}_o/\bar{\rho}_o} \end{aligned} \tag{8}$$

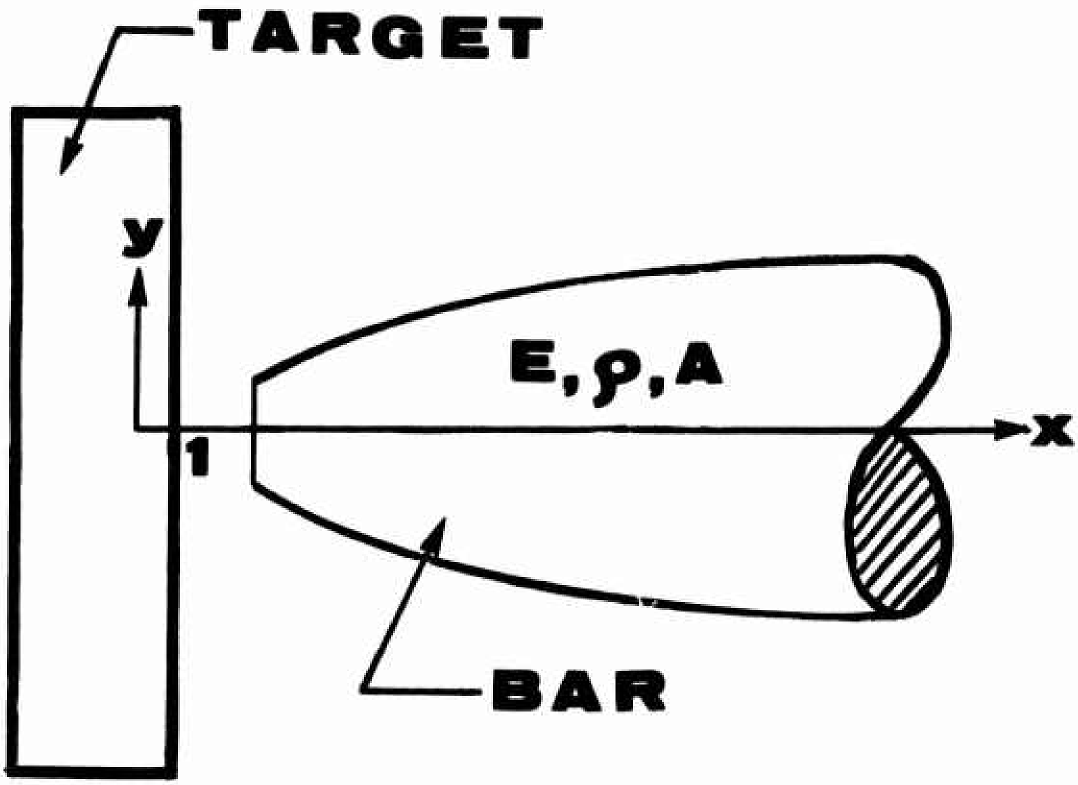

Figure 1. Projectile Target Impact; Coordinate System

The Equations (1 to 6) are transformed into

$$\frac{\partial \sigma}{\partial x} + \frac{A'}{A}\sigma = \rho \frac{\partial^2 u}{\partial t^2} \tag{9}$$

$$\sigma = E \frac{\partial u}{\partial x} \tag{10}$$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} + c^2\left[\frac{A'}{A} + \frac{E'}{E}\right]\frac{\partial u}{\partial x} \tag{11}$$

$$c^2 = E/\rho \tag{12}$$

$$u = 0 = \frac{\partial u}{\partial t} \text{ for: } 1 \leq x \leq \infty,\ t \leq 0 \tag{13}$$

$$\frac{\partial u}{\partial x} = H(t) \text{ at } x = 1 \tag{14}$$

It is convenient to note that Equation (11) is a linear wave equation with variable coefficients and is expressed only in terms of displacement. Thus provided the wave speed, c, is real, the equation is of the hyperbolic type and admits of propagating wave solutions.

Solution Technique

It is Equation (11), subject to Equations (13) and (14) which is to be solved. In general, two types of solutions are available: numerical and analytic. A numerical solution, such as the finite difference form of the method of characteristics, can be obtained for virtually any continuous choice of E, ς .[17] An analytic solution can be obtained, however, only when the choice of these same quantities leads to a Sturm-Louiville problem. Thus choices for E, ς , and A must render Equation (11) amenable to solution in terms of such orthogonal functions as the Legendre, Bessel, trigonometric or hypergeometric functions. Even then only one function, the Bessel function, allows an analytic solution of Equation (11) to appear in closed form in terms of elementary functions. It is this type of solution which is demonstrated here.

Using References 1, 2, 5, 10, 17, 18, the following choices for E and ς are made

$$E = x^{\alpha} \tag{15}$$

$$\varsigma = x^{\lambda} \tag{16}$$

and

$$\alpha \gtreqless 0 \, ; \; \lambda \gtreqless 0 \tag{17}$$

where α and λ are constants. As will be shown such a choice allows solutions of Equation (11) to be obtained in closed form.

---

*See References 1 through 18.

Equation (11) becomes

$$\frac{\partial^2 u}{\partial x^2} + \frac{1}{x}(\alpha + k)\frac{\partial u}{\partial x} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \tag{18}$$

where c^2 now has the special form

$$c^2 = x^{\alpha - \lambda} = x^s \tag{19}$$

and s is set equal to $\alpha - \lambda$. In Equations (18) and (19), it has been assumed that the area is represented as a power law in the form

$$A(x) = x^k \tag{20}$$

which represents a solid of revolution about the x-axis as shown in Figure 1. Several typical projectile shapes contained in this analysis are shown in Table 1.

TABLE 1.

Typical Projectiles Incorporated in Equation 20

Value of k	Type of Projectile
0	Right circular cylinder
1	Ogival (truncated paraboloid)
2	Truncated cone

It is appropriate to note here that equations similar to Equation (18) govern solutions to other types of elastic wave propagation germane to munitions problems. These are:

1. An axially impacted membrane shell with variable density and modulus.[2]

2. Axial shearing of the internal diameter of a cylindrical bore in an elastic half space.[17]

3. A spherical blast wave in an inhomogeneous medium.[18]

4. A cylindrical blast wave in an inhomogeneous medium.[18]

Problem 1 is applicable to a cartridge case which may be modeled during extraction as a truncated conical membrane shell with variable properties. The case receives axial impulses from both burning propellant and vibratory contact with the bolt. Problem 2 describes, approximately, the radial propagation of axial shear waves in a target plate due to projectile penetration at normal obliquity. This is part of the far field shear motion generated at the shear interface between projectile and plate. Problems 3 and 4 describe the far-field response of an inhomogeneous medium (such as the earth) to detonation of confined or buried explosive.

The solution of Equation (18) satisfying the initial conditions, Equation (13), is obtained by taking the Laplace transform of Equation (18) and employing Equation (13). The Laplace Transform of Equation (18) is

$$\frac{d^2\bar{U}}{dx^2} + \frac{(a+k)}{x}\frac{d\bar{U}}{dx} - p^2 x^{-s}\,\bar{U} = 0 \tag{21}$$

where

$$\bar{U}(x,p) = \int_{0^+}^{\infty} \exp(-pt)\cdot u(x,t)\,dt$$

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*See References 2 through 18.

The solution of Equation (21) is given in Reference 19 as

$$U(x,p) = x^{a}\left[A(p)\, I_n\!\left(\frac{p}{\beta}x^{\beta}\right) + B(p)\, K_n\!\left(\frac{p}{\beta}x^{\beta}\right)\right] \qquad (22)$$

$$\text{for } \beta \neq 0$$

Here p is the transform parameter, A(p) and B(p) are (as yet) unspecified constants, and U(x, p) is the transform of u(x, t). Capital letters will henceforth represent the transform of the corresponding lower case variable. $I_n(z)$ and $K_n(z)$ are modified Bessel Functions of the order n and of the first and second kinds, respectively. For convenience, a and β are defined as

$$a = \frac{1-(R+\alpha)}{2} \qquad (23)$$

$$\beta = \frac{2-S}{2}$$

and

$$n = a/\beta = [1-(R+\alpha)]/[2-S] \qquad (24)$$

or

$$n = -a/\beta = [1-(R+\alpha)]/[S-2]$$

Following convention, that is considering β to be greater than zero, we choose $A(p)=0$ in Equation (22) due to the unbounded behavior of $I_n(z)$ as z approaches infinity for $\beta > 0$. No attempt to establish this assumption rigorously from the usual "boundedness at infinity" arguments of References 2, 17, 18 for boundedness on either displacement[2] or stress[17, 18] is implied. For the case of $\beta > 0$, the boundary condition[14] then determines $B(p)$ as

$$-B(p) = 1/\left[p^2 K_{n+1}\!\left(\frac{p}{\beta}\right)\right] \qquad (25)$$

---

* See References 2 through 19.

and the general transformed solution for displacement is then obtained from Equations (22) and (25). Similarly the transformed strain dU/dx, stress Σ and particle velocity V are, respectively,

$$\frac{dU}{dx} = -B(p)\left[p K_{n+1}\left(\frac{p}{\beta} x^{\beta}\right)\right] x^{(\alpha+\beta-1)} \tag{26}$$

$$\Sigma = \frac{dU}{dx} x^{\alpha} \tag{27}$$

and

$$V = pU \tag{28}$$

In deriving Equation (25) it has been assumed that n takes on the negative sign in Equation (24). This choice will be sufficiently general for what follows. The restriction of $\beta > 0$ will be discussed subsequently.

As shown in References[17,18] when n becomes half an odd integer, the Equations (22), (26), (27) and (28) may be inverted into closed form solutions. This is easily noted from the fact that when n is half an odd interger, i.e.,

$$n = +\alpha/\beta = j + \frac{1}{2} \quad ; \qquad j = 0, \pm 1, \pm 2, \ldots \tag{29}$$

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[17] P.C. Chou and P.F. Gordon, "Radial Propagation of Axial Shear Waves in Nonhomogeneous Elastic Media", Journ. Acoust. Soc. Amer., Vol. 42, No. 1, pp 36-41, 1967.

[18] E. Sternberg and J. Chakravorty, "On the Propagation of Shock Waves in a Nonhomogeneous Elastic Medium", J. A. M., Vol. 26, Trans. ASME, Vol. 81, Series E, 1959, pp 528-536.

the Bessel functions become elementary finite series of the form[17,18]

$$K_{(j+1/2)}(z) = \left(\frac{\pi}{2z}\right)^{1/2} \cdot \exp(-z) \cdot \sum_{r=0}^{j} \frac{(j+r)!}{r!(j-r)!(2z)^r} \tag{30}$$

and

$$K_{-n}(z) = K_n(z) \quad \text{any } n \tag{31}$$

which have the immediate benefit that the infinite series for U, Σ, V become the ratio of two finite polynomial series in ρ. Such expressions are readily invertable.

It is seen from the above expressions that corresponding to each j there are an infinite number of α, λ and k which satisfy each term of Equation (28). In a problem in which α, λ, and k are specified it is immaterial, by Equation (30), whether the negative or positive sign of Equation (23) is chosen. However the form of Equations (24) to (27) was derived using the negative sign; thus the negative sign should be used in practice.

In order to demonstrate the range of solutions possible for various shaped projectiles we consider three examples: cylindrical, ogival and conical. From Table II these correspond to the area parameter k, having values of 0, 1 and 2 respectively. Equation (29) is used as a generating function for all closed form solutions by allowing j to take on every integer and zero. Specifically, select projectiles of variable

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[17] P.C. Chou and P.F. Gordon, "Radial Propagation of Axial Shear Waves in Nonhomogeneous Elastic Media", Jour. Acoust. Soc. Amer., Vol. 42, No. 1, pp 36-41, 1967.

[18] E. Sternberg and J. Chakravorty, "On the Propagation of Shock Waves in a Nonhomogeneous Elastic Medium", J. A. M. Vol. 26, Trans. ASME, Vol. 81, Series E, 1959, pp 528-536.

elastic modulus α , but constant density $\lambda = 0$. The generating functions, Equation (29), have the form:

$$\text{cylinder} \quad = 4j/(2j - 1) \tag{32}$$

$$\text{ogive} \quad = 2(1+2j)/(2j-1) \tag{33}$$

$$\text{cone} \quad = 4(1 + j)/(2j - 1) \tag{34}$$

TABLE II

Closed Form Solutions for Various Shaped Projectiles: $E = E_0 X^{\alpha}$

Cylindrical k = 0		Ogival k = 1		Conical k = 2	
j	α	j	α	j	α
0	0[a]	0	-2	0	-4
1	4	1	6	1	8
2	8/3	2	10/3	2	4
3	12/5	3	14/5	3	16/5
.	.	.	.	.	.
.	.	.	.	.	.
.	.	.	.	.	.
∞	2	∞	2	∞	2
-1	4/3	-1	2/3	-1	0[b]
-2	8/5	-2	6/5	-2	4/5
-3	12/7	-3	10/7	-3	8/7
.	.	.	.	.	.
.	.	.	.	.	.
.	.	.	.	.	.
$-\infty$	2	$-\infty$	2	$-\infty$	2

[a] Homogeneous cylinder

[b] Homogeneous cone

A few values of α from these generating functions are given in Table II. It should be noted that the usual homogeneous solution for the cylinder and cone are special cases. An interesting fact, which may be deduced from Equations (32) to (34) or Table II, is that in the limit as j goes to infinity, negatively or positively, the limiting closed form solution is $\alpha = 2$ or $\beta = 0$. Note that for $\beta = 0$ Equation (22) is singular.

Two detailed examples of the present solution technique for the case of a paraboloidal projectile with a variable modulus are now presented. As stated earlier, the choice of $k = 1$, $\lambda = 0$, but α variable leads to the formulation of Reference 17. Two closed form solutions not presented in that reference follow. These correspond to the cases of $\alpha = 6/5$ and $\alpha = -2$. Following exactly the procedure of Reference 17, using Equations (22) to (31), we obtain the solutions in transform space for the displacement as

for $\underline{\alpha = 6/5}$ $\quad (n = 3/2,\ \beta = 2/5)$

$$U(x,p) = -x^{(\beta+\alpha)/2} \left\langle \exp\left[\left(\frac{1-x^{\beta}}{\beta}\right)p\right]\right\rangle \left[\left(\frac{1}{p^2} + \frac{\beta}{p^3 x^{\beta}}\right)\left(1 + \frac{3\beta}{p} + \frac{3\beta^2}{p^2}\right)^{-1}\right] \tag{35}$$

for $\underline{\alpha = -2}$ $\quad (\beta = 2,\ n = -1/2)$

$$U(x,p) = p^{-2} \cdot x^{-(\alpha+\beta)/2} \left\langle \exp\left[p\left(\frac{1-x^{\beta}}{\beta}\right)\right]\right\rangle \tag{36}$$

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[17] P. C. Chou and P. F. Gordon, "Radial Propagation of Axial Shear Waves in Nonhomogeneous Elastic Media", Journ. Acoust. Soc. Amer., Vol. 42, No. 1, pp 36-41, 1967.

The inversion of Equations (35) and (36) requires only a standard transform-pair table and repeated application of the convolution (Faltung) theorem. The final results are:

$$\underline{\alpha = 6/5}$$

$$u(x,t) = -x^{-4/5} \cdot H(t-a_1)\Big\{\exp\left(-\tfrac{3}{5}[t - a_1]\right) \cdot \left(\left[\tfrac{5\sqrt{3}}{3} - \tfrac{5\sqrt{3}}{6} x^{-2/5}\right] \cdot [\sin\gamma(t-a_1)] + -\tfrac{5}{6} x^{-2/5} \cdot \cos\gamma(t-a_1)\right) + 5/6\, x^{-2/5}\Big\} \tag{37}$$

and

$$\sigma(x,t) = x^{6/5} H(t-a_1)\Big\{\exp\left(-\tfrac{3}{5}[t - a_1]\right) \cdot \left(\left[2x^{-9/5} - x^{-11/5} - x^{-7/5}\right] \cdot [\sin\gamma(t-a_1)] + \left[x^{-7/5} - x^{-11/5}\right] \cdot [\cos\gamma(t-a_1)] + x^{-11/5}\right)$$

where

$$a_1(x) = \tfrac{5}{2}\left(x^{2/5} - 1\right) \; ; \; \gamma = \tfrac{\sqrt{3}}{5}$$

For $\underline{\alpha = -2}$

$$u(x,t) = H(t-a_2)(a_2 - t) \tag{38}$$

$$\sigma(x,t) = H(t-a_2)\, x^{-1}$$

and $$a_2(x) = [x^2 - 1]/2 \tag{39}$$

The stresses in Equations (37) and (38) and the displacement in Equation (37) are displayed graphically in Figures 2 to 4.

As can be seen from Figures 2 to 4 for the case of $\alpha = 6/5$ the displacement and stress are well behaved and remain finite for all values of time. For the somewhat degenerate case of $\alpha = -2$ the stress is well behaved, tending to zero as x increases. For the same problem, however, we note that the displacement at a fixed spacial location goes unbounded with increasing time. This latter case satisfies the condition of $\beta > 0$ and the regularity of stress given by Equation (40) which will be presented later. It is concluded, therefore, that even when the wave speed is bounded and the stress well behaved no assurance that the displacement is bounded may be implied.

Wave Front Information

By virtue of the application of the discontinuous (step) stress at $x = 1$, and the hyperbolicity of Equation (18), it may be expected to propagate waves in which the stress, strain and particle velocity are discontinuous, while the displacement remains continuous. Such elastic discontinuities are well understood.[20 to 22] Note[22] that the "first characteristic" or locus in $x - t$ space of the leading wave front is given by

$$t_a(x) = \begin{cases} \dfrac{2}{2-s}\left[x^{\left(\frac{2-s}{2}\right)} - 1\right], & s \neq 2 \\ \ln x, & s = 2 \end{cases} \tag{40}$$

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*See Reference 20 to 22.

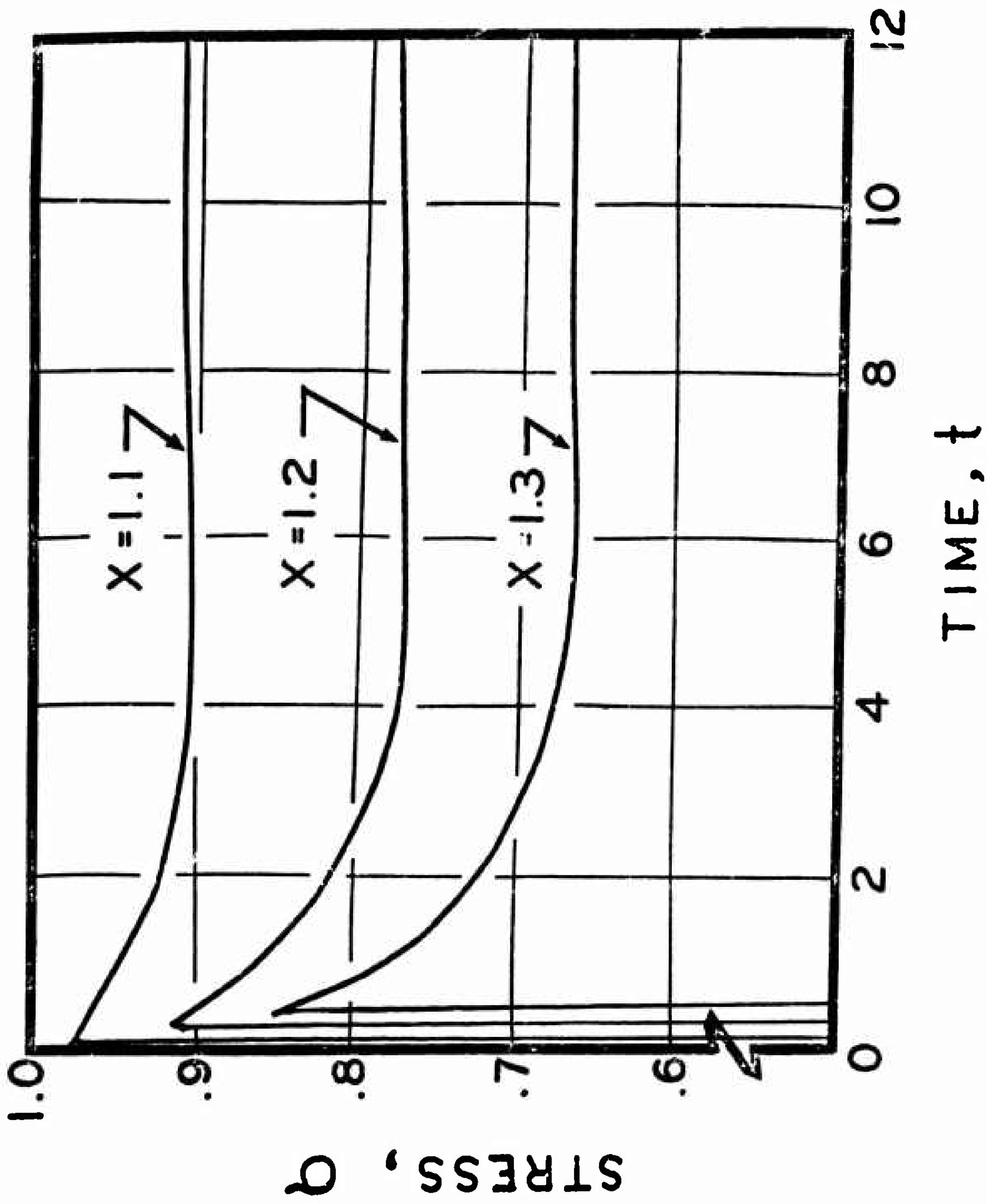

Figure 2. Stress as a Function of Time at Several Locations in an Ogival Projectile: $\alpha = 6/5$

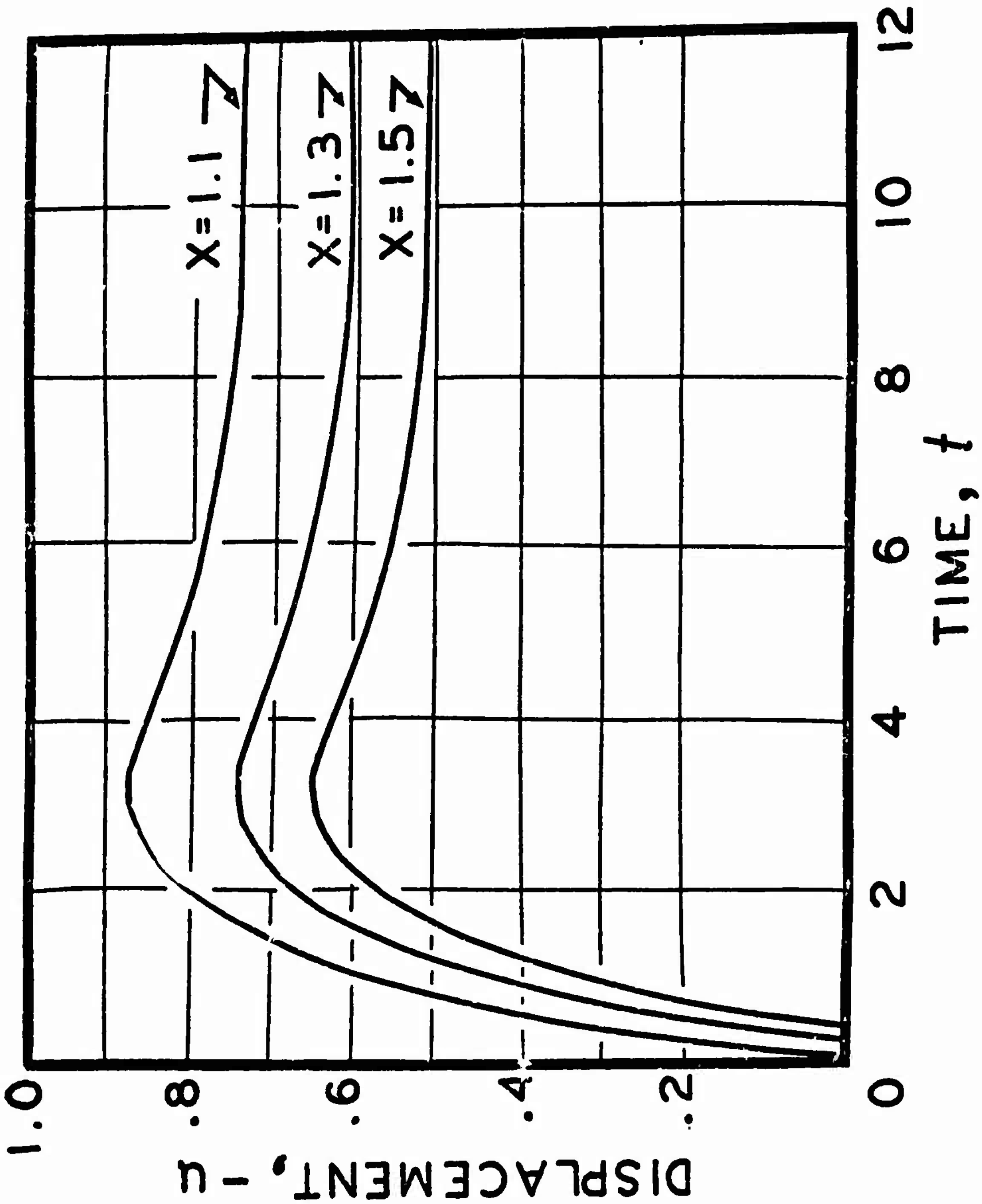

Figure 3. Displacement as a Function of Time at Several Locations in an Ogival Projectile: $\alpha = 6/5$

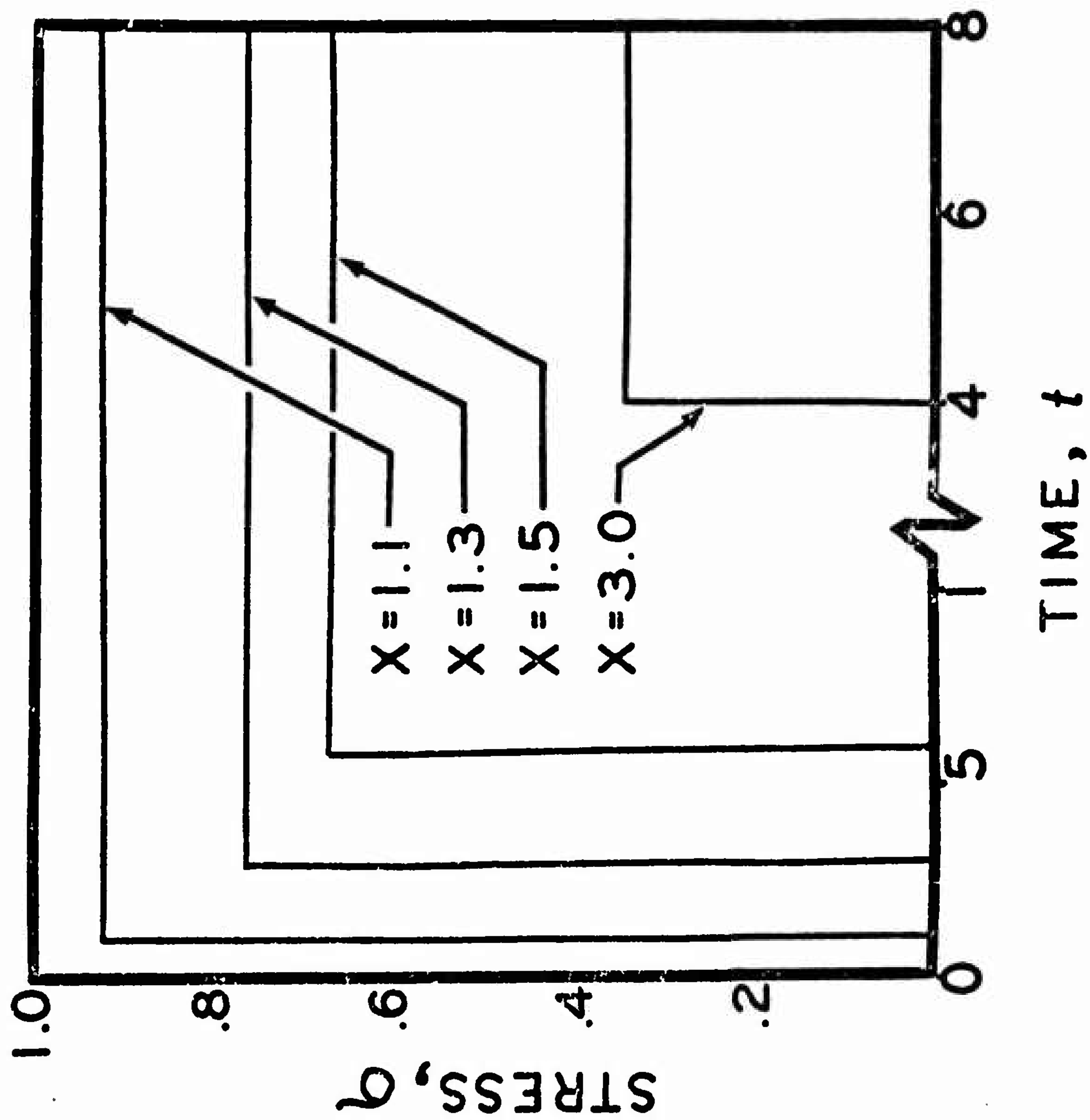

Figure 4. Stress as a Function of Time at Several Locations in an Ogival Projectile: $\alpha = -2$

where t_a, physically, is the arrival time of the leading wave at the location x. The graph of Equation (40) is shown in Figure 5. As x is allowed to approach infinity in Equation (40) then the time of arrival at $x=\infty$ is given by

$$t_\infty = \begin{cases} \infty & , \; s \leq 2 \\ -2/(2-s) & , \; s > 2 \end{cases} \tag{41}$$

If $s>2$, or $\beta<0$, the wave speed increases without bound, and the wave reaches the end of the infinite rod in the finite time as indicated by Equation (41). Such a situation will, for the present report, be disregarded on obvious physical grounds. Thus, as stated earlier, only the case of $s < 2$ (or $\beta > 0$) will be treated. It should be noticed, however, that the case discarded here, $\beta < 0$, has been successfully treated in earlier works.[17, 18] It is suspected, but unproven, that the solutions obtained by inverting Equations (22) to (28) are mathematically correct for certain time intervals in the case of $\beta < 0$. This opinion is supported in part by References 17, 18 and, apparently, disagreed with in Reference 2.

In the case of discontinuous boundary data, discontinuities or "jumps" in the first derivatives of displacement across the leading wave may be expressed analytically.[22] If it is assumed that the projectile is initially unstressed and motionless ahead of the leading wave then the values of stress, strain and particle velocity immediately behind the leading wave are given by

$$\sigma = x^{(\lambda+2\alpha-2k)/4} \tag{42}$$

$$\frac{\partial u}{\partial x} = x^{(\lambda-3\alpha-2k)/4} \tag{43}$$

and

$$-\frac{\partial u}{\partial t} = x^{(-\alpha-\lambda-2k)/4} \tag{44}$$

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*See References 2 through 22.

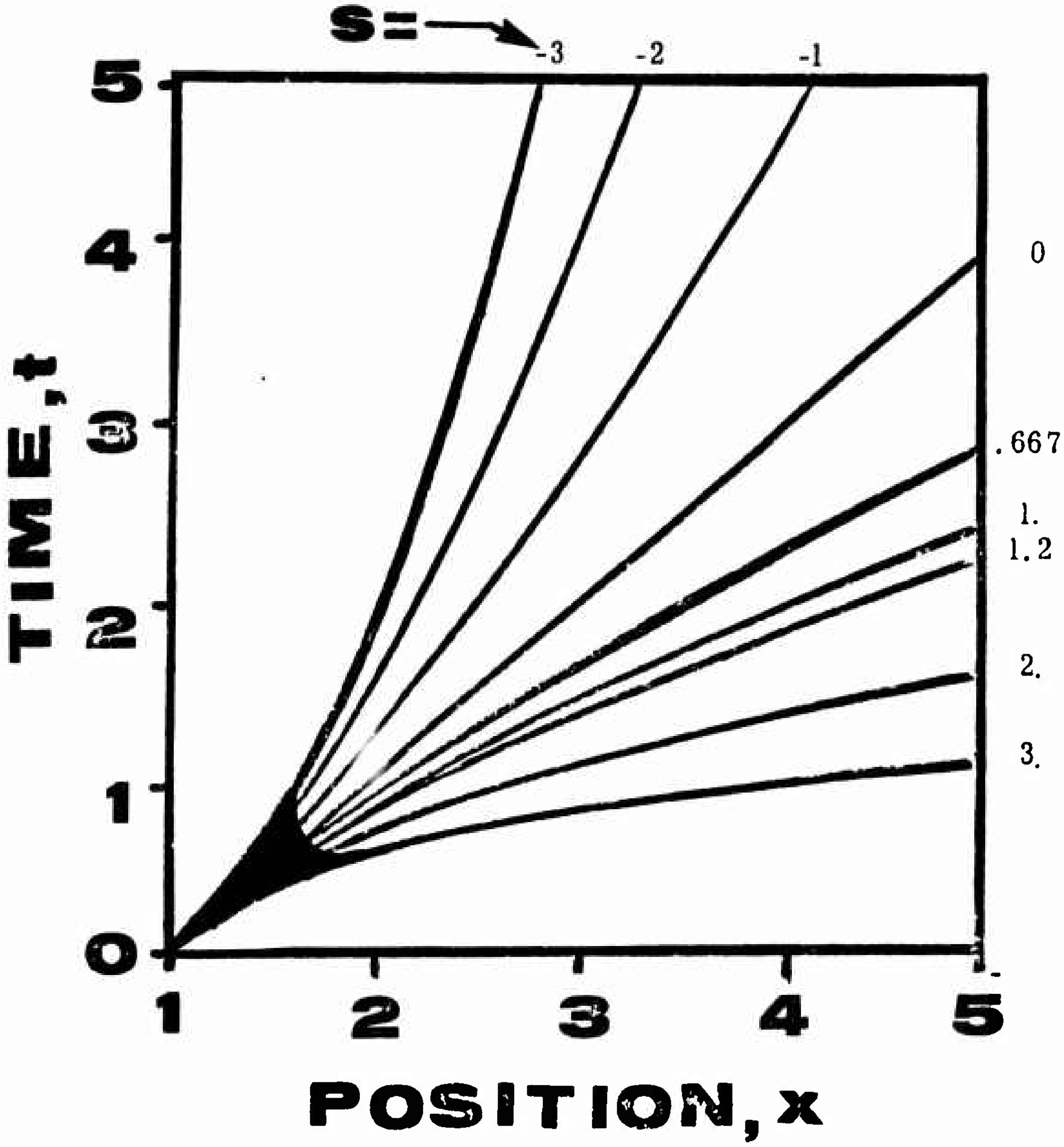

Figure 5. Time of Arrival, t, as a Function of x for Various Values of S. (Equation 40)

provided the boundary condition in the Equation (14) is invoked. The expressions in the Equations (42) to (44) apply only at the time t_a, in Equation (40), and only for $\beta \geq 0$. *

An application of the preceding wavefront information theory is given in Table III. This shows the instantaneous values of stress, strain and strain energy for the elastic projectiles of Table II at the wavefront. These projectiles have a variable modulus, $E = EoX^{\alpha}$, and constant density, $\lambda = 0$. (The coordinate x is the location of the wavefront corresponding to the arrival time, t_a, of Equation (40)). In a homogeneous structure the stress, strain and strain energy are either constant or are decreasing as the wave proceeds down the projectile. This effect, constancy or decrease of the wave strength, is a form of geometric (areal) attenuation. As the crossectional area is either constant (cylinder) or increases with x (ogive, cone) the stress, strain and strain energy remain constant (cylinder) or decrease (ogive, cone). However, in the same structures with inhomogeneity the stress, strain and strain energy may be markedly different as shown in Table III. This could be important in estimating failure. Some modern failure theories postulate that failure is related to strain energy. Failure in these three homogeneous, infinite projectiles (if it can occur) will occur at the impact surface (x = 1), which is the position of maximum strain energy. For an inhomogeneous structure the strain energy may actually be greater at some point other than the impact location. Thus it cannot be assumed that failure can for such structures need be localized only to the impact surface.

Information of the type found in expressions in the Equations (42) to (44) may be found for other classes of inhomogeneity than discussed

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* The question of the applicability of Equations (42) to (44) for $\beta < 0$ or $\beta > 2$ is unresolved in the present report. It was found in Reference 17 that unique solutions for $\beta > 2$ were obtainable by the method of characteristics only for certain restricted regions in x - t space. The Equations (42) to (44) arise from considerations in the method of characteristics.

TABLE III

Distribution of Stress, Strain and Strain Energy at the Wavefront for Various Projectiles

INHOMOGENEOUS Projectile			
	Cylindrical	Ogival	Conical
Stress	$x^{\alpha/4}$	$x^{(\alpha-2)/4}$	$x^{(\alpha-4)/4}$
Strain	$x^{-3\alpha/4}$	$x^{-(3\alpha+2)/4}$	$x^{-(3\alpha+4)/4}$
Strain Energy	$x^{-\alpha/2}$	$x^{-(\alpha+2)/2}$	$x^{-(\alpha+4)/2}$
HOMOGENEOUS Projectile			
	Cylindrical	Ogival	Conical
Stress	1	$x^{-1/2}$	x^{-1}
Strain	1	$x^{-1/2}$	x^{-1}
Strain Energy	1	x^{-1}	x^{-2}

here. For the general formulation the reader is referred to References (20) to (22).

As can be seen from Equation (42) unless

$$\lambda + \alpha \leq 2k \tag{45}$$

is satisfied, the usual regularity condition, i.e.,

$$\lim_{x \to \infty} \sigma(x,t) = 0, \quad t \geq 0 \tag{46}$$

cannot be satisfied along the wave front. Previous, but more restrictive, investigations[17,*] have shown that the type of condition deduced in Equation (45) is equivalent to choosing $\beta \geq 0$. In the present paper, however, by recalling Equation (23) one sees that $\beta \geq 0$ does not necessarily imply Equation (45). It would appear that even for $\beta > 0$ the regularity condition, Equation (46), will not necessarily be satisfied.

SUMMARY

The principal result of this study was the successful application of two mathematical techniques to the solution of the problem of the linear elastic wave propagation in an inhomogeneous bar. The first technique, the Laplace transform, provided closed form expressions for the states of dynamic stress, strain and displacement in the bar. The modulus, density and crossectional area were simultaneously varied with location in the rod. The second technique, based on the theory of discontinuities, provided simple expressions for the arrival time, dynamic stress, strain and strain energy at the wave front in the projectile. The values of dynamic stress and strain obtained from both methods were found to be identical.

*See References 17 through 22.

The first technique was applied to a paraboloidal projectile with constant density and two types of variable modulus, one increasing and one decreasing with distance from the point of impact. The results for an increasing modulus projectile ($\alpha = 6/5$) in Figures 2 and 3 showed that the stress and displacement decreased monotonically with distance from the impact end for all times. The peak values of stress at every location occurred at the time of wave arrival, while the peak displacements occurred a short time after wave arrival. It was found that while the displacement was continuous for all x and t, the stress was singularly discontinuous at the wavefront. The results for a decreasing modulus projectile ($\alpha = -2$) in Figure 4 and Equation (38) showed again that stress and displacement monotonically decrease with distance. The stress, however, jumped discontinuously at the wavefront from zero to a steady state value and remains constant thereafter. This lack of transient behavior was completely different from the case of $\alpha = 6/5$, in which the steady state values of stress and displacement are preceded by a transient behavior.

In addition, the displacement for $\alpha = -2$ has no finite steady state value, it increases indefinitely with time at every location. It was also found that the peak dynamic stress is greatest in the $\alpha = 6/5$ projectile, but that the steady state stress was greatest in the $\alpha = -2$ projectile. At a location .3 (nondimensional) units from the impact end ($x = 1.3$) the peak values of stress for $\alpha = 6/5$ and $\alpha = -2$, respectiv ly, are 85% and 76% of the applied impact stress. The corresponding values of the final steady state, or static, stress are, respectively 67% and 76%.

Results of the application of discontinuity theory (Table III) showed clearly that for a homogeneous projectile the stress, strain and strain-energy at the wavefront are either constant (right cylinder) or decrease (ogive, cone) as the wave proceeds down the rod. These results were due to areal attenuation, i. e., the constancy (cylinder) or increase (ogive, cone) of area over which the stress acted. For the same, but inhomogeneous, projectiles the results are markedly different. An increase or decrease in stress, strain and energy at the wavefront depended upon the amount of inhomogeneity (value of α). For example, in an ogival projectile the stress, strain or strain energy <u>increased</u> at the wavefront if the values of α were greater than 2, less than -2/3 or less than -2, respectively.

Based on these results, the following conclusions are drawn:

1. The characteristics of linear elastic wave propagation in inhomogeneous long rod penetrators are markedly different from those found in homogeneous penetrators. Specifically, it is found that the wave speed and arrival time are functions of position and that the states of stress and strain depend upon the amount of inhomogeneity.

2. If material failure is assumed to depend only on strain energy, then failure may occur at locations other than the impact surface.

3. Certain classes of inhomogeneity led to wave speeds such that the wave front reaches infinity in a finite time. These cases, while mathematically correct, are physically impossible and represent restrictions on the applicability of the theory.

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